Localized electromagnetic and weak gravitational fields in the source-free space

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Localized electromagnetic and weak gravitational time-harmonic fields in the source-free space are treated using expansions in plane waves. The presented solutions describe fields having a very small (about several wavelengths) and clearly defined core region with maximum intensity of field oscillations. In a given Lorentz frame L, a set of the obtained exact time-harmonic solutions of the free-space homogeneous Maxwell equations consists of three subsets (storms, whirls, and tornados), for which time average energy flux is identically zero at all points, azimuthal and spiral, respectively. In any other Lorentz frame L', they will be observed as a kind of electromagnetic missile moving without dispersing at speed $V<c$. The solutions that describe finite-energy evolving electromagnetic storms, whirls, tornados, and weak gravitational fields with similar properties are also presented. The properties of these fields are illustrated in graphic form.

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I. INTRODUCTION

In the beginning of the 1980s, Brittingham [1] proposed the problem of searching for specific electromagnetic waves—focus wave modes—having a three-dimensional pulse structure, being nondispersive for all time, and moving at light velocity in straight lines. Some packetlike solutions have been presented [1–3], but it seems likely that finite-energy focus wave modes cannot exist without sources [3–5]. In 1985, Wu [5] proposed the concept of electromagnetic missiles moving at light velocity and having a very slow rate of decrease with distance.

In our previous publications [6–9], we have investigated linear fields defined by a given set of orthonormal scalar functions on either a two-dimensional or a three-dimensional manifold. The suggested approach [7] makes it possible to obtain families of orthonormal beams and other specific exact solutions of wave equations. It can be applied to any linear field, such as electromagnetic waves in free space, isotropic, anisotropic, and bianisotropic media, elastic waves in isotropic and anisotropic media, sound waves, weak gravitational waves, etc. As examples we have presented electromagnetic orthonormal beams and three-dimensional standing waves in free space and isotropic media [6–9], including the chiral ones [9].

By forming convenient functional bases for complex fields, the orthonormal beams provide a means to generalize the free-space techniques [10–14] for characterizing complex media as well as the covariant wave-splitting technique [15] to the case of incident beams. The three-dimensional standing waves give a unique global description of the complex medium under study, which is supplementary to the eigenwave description. Even in free space they possess very interesting properties [6–9].

In this paper, we present unique solutions that describe localized electromagnetic and weak gravitational fields with possible applications in physics and astrophysics. The plan of the paper is as follows. In Sec. II we sketch the basics of plane-wave superpositions defined by a given set of orthonormal scalar functions on a real manifold. Localized electromagnetic and gravitational time-harmonic fields are treated in Secs. III–V. Moving and evolving fields are briefly discussed in Sec. VI. Concluding remarks are made in Sec. VII.

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II. BASIC EQUATIONS

A. Fields defined by orthonormal functions on a real manifold

Let (u_n) be a set of complex scalar functions on a real manifold $\mathcal{B}_u$, satisfying the orthogonality conditions

$$\langle u_m | u_n \rangle \equiv \int_{\mathcal{B}_u} u_m^*(b) u_n(b) d\mathcal{B} = \delta_{mn}, \tag{2.1}$$

where $d\mathcal{B}$ is the infinitesimal element of $\mathcal{B}_u$, u_m^* is the complex conjugate function to u_m, and δ_{mn} is the Kronecker δ function. Let us consider a plane-wave superposition (termed below the ''beam'' for the sake of brevity)

$$\mathbf{W}_n(\mathbf{x}) = \int_{\mathcal{B}_u} e^{i\mathbf{x}\cdot\mathbf{K}(b)} u_n(b)\nu(b)\mathbf{W}(b) d\mathcal{B}$$
$$= \int_{\mathcal{B}} e^{i\mathbf{x}\cdot\mathbf{K}(b)} u_n(b)\nu(b)\mathbf{W}(b) d\mathcal{B}, \tag{2.2}$$

where $\mathbf{x}$ and $\mathbf{K}$ are the four-dimensional position and wave vectors, and $\mathcal{B}$ is a subset of $\mathcal{B}_u$ with nonvanishing values of function $\mathbf{W}' = \nu(b)\mathbf{W}(b)$. Here, $\mathbf{W}$ can be any of the following quantities: the electric (magnetic) field vector $\mathbf{E}$ ($\mathbf{B}$), the six-dimensional vector col($\mathbf{E}$,$\mathbf{B}$), the four-dimensional field tensor F (for electromagnetic waves), the small variation h of the metric tensor (for weak gravitational waves). A set of plane waves forming the beam (beam base) is specified by functions $\mathbf{K}=\mathbf{K}(b)$ and $\mathbf{W}=\mathbf{W}(b)$, whereas a beam state is given by a complex scalar function $\nu=\nu(b)$.

There are four key elements defining the properties of these beams: the set of functions $u_n=u_n(b)$, the beam manifold $\mathcal{B}$, the beam base, and the beam state function ν

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$=\nu(b)$. By setting these elements in various ways, one can obtain a multitude of interesting fields [6–9], among them orthonormal beams satisfying the condition

$$\langle \mathbf{W}_m|Q|\mathbf{W}_n\rangle \equiv \int_{\sigma_0} \mathbf{W}_m^\dagger(\mathbf{x}) Q \mathbf{W}_n(\mathbf{x}) d\sigma_0 = N_Q \delta_{mn}, \tag{2.3}$$

where σ_0 is either a two-dimensional or a three-dimensional manifold, Q is some Hermitian operator, and $\mathbf{W}_m^\dagger(\mathbf{x})$ is the Hermitian conjugate of $\mathbf{W}_m(\mathbf{x})$.

Time-harmonic beams with two-dimensional manifold $\mathcal{B}$ can be written as [7]

$$\mathbf{W}_n(\mathbf{r},t) = e^{-i\omega t}\int_{\mathcal{B}} e^{i\mathbf{r}\cdot\mathbf{k}(b)} u_n(b)\nu(b)\mathbf{W}(b) d\mathcal{B}, \tag{2.4}$$

where $\mathbf{r}$ and $\mathbf{k}$ are the three-dimensional position and wave vectors in a Lorentz frame L with basis $(\mathbf{e}_i)$, i.e., $\mathbf{x}=\mathbf{r}+ct\mathbf{e}_4$ and $\mathbf{K}=\mathbf{k}+(\omega/c)\mathbf{e}_4$. Here, c is the velocity of light in vacuum, t is the time in L, ω is the angular frequency, $\mathbf{e}_i^2=1$, $i=1,2,3$, and $\mathbf{e}_4^2=-1$.

It can be shown [7] that these beams become orthonormal, provided the following conditions are met.

(i) σ_0 is a plane with unit normal $\mathbf{q}$, passing through the point $\mathbf{r}=0$.

(ii) The tangential component

$$\mathbf{t}(b) = \mathbf{k}(b) - \mathbf{q}[\mathbf{q}\cdot\mathbf{k}(b)] \tag{2.5}$$

of $\mathbf{k}(b)$ is real for all $b\in\mathcal{B}$, and the mapping $b\mapsto\mathbf{t}(b)$ is one-one (injective).

(iii) $\mathcal{B}=\mathcal{B}_u$, and the function $\nu(b)$ is given by

$$\nu(b) = \frac{1}{2\pi}\sqrt{\frac{N_Q J(b)}{g(b)\mathbf{W}^\dagger(b) Q \mathbf{W}(b)}}. \tag{2.6}$$

Here, $J(b)=D(t^j)/D(\xi^i)$ is the Jacobian determinant of the mapping $b\mapsto\mathbf{t}(b)$, calculated in terms of the local coordinate systems $(\xi^i, i=1,2)$ and $(t^j, j=1,2)$, and $d\mathcal{B}=g(b)d\xi^1 d\xi^2$.

In this paper, we treat electromagnetic fields in free space and weak gravitational fields, defined by the spherical harmonics $Y_j^s(\theta,\varphi)$ as

$$\mathbf{W}_j^s(\mathbf{r},t) = e^{-i\omega t}\int_0^{2\pi} d\varphi \int_{\theta_1}^{\theta_2} e^{i\mathbf{r}\cdot\mathbf{k}(\theta,\varphi)} Y_j^s(\theta,\varphi) \times \nu(\theta,\varphi)\mathbf{W}(\theta,\varphi)\sin\theta\, d\theta, \tag{2.7}$$

where

$$Y_l^m(\theta,\varphi) = N_{lm} P_l^{|m|}(\cos\theta) e^{im\varphi}, \tag{2.8}$$

$$N_{lm} = \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}, \tag{2.9}$$

and $P_l^m(\cos\theta)$ are the spherical Legendre functions [16,17]. For these fields, $\mathcal{B}_u$ is a unit sphere ($\mathcal{B}_u=S^2$), $\mathcal{B}$ is its zone with $\theta\in[\theta_1,\theta_2]$ and $\varphi\in[0,2\pi]$, and $d\mathcal{B}=\sin\theta\, d\theta\, d\varphi$.

By using the Rayleigh formula [16], the fields under consideration can be expanded into a series as [7]

$$\mathbf{W}_j^s(\mathbf{r},t) = e^{-i\omega t}\sum_{l=0}^{+\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{r}})\mathbf{W}_l^m, \tag{2.10}$$

where $k=2\pi/\lambda=\omega/c$, $j_l(kr)$ are the spherical Bessel functions [16,17], $Y_l^m(\hat{\mathbf{r}})=Y_l^m(\gamma,\psi)$, and

$$\hat{\mathbf{r}} = \mathbf{r}/r = \sin\gamma(\mathbf{e}_1\cos\psi + \mathbf{e}_2\sin\psi) + \mathbf{e}_3\cos\gamma. \tag{2.11}$$

The coordinate independent vector coefficients $\mathbf{W}_l^m$ completely characterize these fields [7].

B. Beam bases

To set the beam base of fields $\mathbf{W}_j^s(\mathbf{r},t)$ (2.7), it is necessary to specify propagation directions [unit wave normals $\hat{\mathbf{k}}\equiv\mathbf{k}/k=\hat{\mathbf{k}}(\theta,\varphi)$] and polarizations [normalized vector amplitudes $\mathbf{W}(\theta,\varphi)$] of all partial plane waves. The former can be set both for electromagnetic and weak gravitational fields by

$$\hat{\mathbf{k}}(\theta,\varphi) = \sin\theta'(\mathbf{e}_1\cos\varphi' + \mathbf{e}_2\sin\varphi') + \mathbf{e}_3\cos\theta', \tag{2.12}$$

where $\theta'=\theta'(\theta,\varphi)$ and $\varphi'=\varphi'(\theta,\varphi)$ are some given functions. In this paper, we restrict our consideration to beams with

$$\theta'=\kappa_0\theta, \quad \varphi'=\varphi, \tag{2.13}$$

where κ_0 is some real parameter, $0<\kappa_0\leqslant 1$. Each of these beams comprises plane waves with wave normals $\hat{\mathbf{k}}$ lying in the same solid angle $\Omega=2\pi(\cos\kappa_0\theta_1-\cos\kappa_0\theta_2)$.

To set the amplitude functions, it is convenient to use the unit basis vectors

$$\mathbf{e}_r(\theta',\varphi) = \sin\theta'(\mathbf{e}_1\cos\varphi + \mathbf{e}_2\sin\varphi) + \mathbf{e}_3\cos\theta', \tag{2.14a}$$

$$\mathbf{e}_{\theta'}(\theta',\varphi) = \cos\theta'(\mathbf{e}_1\cos\varphi + \mathbf{e}_2\sin\varphi) - \mathbf{e}_3\sin\theta', \tag{2.14b}$$

$$\mathbf{e}_\varphi(\varphi) = -\mathbf{e}_1\sin\varphi + \mathbf{e}_2\cos\varphi. \tag{2.14c}$$

1. Amplitude functions for electromagnetic fields

To treat electromagnetic beams in free space, we set

$$\mathbf{W} = \begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix}, \quad Q = \frac{c}{16\pi}\begin{pmatrix}0 & -\mathbf{q}^\times\\ \mathbf{q}^\times & 0\end{pmatrix}, \tag{2.15}$$

where $\mathbf{q}^\times$ is the antisymmetric tensor dual to $\mathbf{q}$($\mathbf{q}^\times\mathbf{E}=\mathbf{q}\times\mathbf{E}$). The normal component of time average Poynting's vector $\mathbf{S}$ can be written as $S_q=\mathbf{q}\cdot\mathbf{S}=\mathbf{W}^\dagger Q\mathbf{W}$. Therefore, the condition $\langle\mathbf{W}_j^s|Q|\mathbf{W}_j^s\rangle=N_Q$ is in fact the normalization to the beam energy flux N_Q through the plane σ_0. We assume below that $\mathbf{q}=\mathbf{e}_3$.

Let us set two amplitude functions by

$$\mathbf{W}(\theta,\varphi)\equiv\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix}=\begin{pmatrix}\mathbf{e}_{\theta'}\\ \mathbf{e}_{\varphi}\end{pmatrix} \tag{2.16a}$$

$$=\begin{pmatrix}\mathbf{e}_{\varphi}\\ -\mathbf{e}_{\theta'}\end{pmatrix}. \tag{2.16b}$$

Beams with the amplitude function $\mathbf{W}$ [Eq. (2.16a)] are formed from plane waves with the meridional orientation of $\mathbf{E}$ and the azimuthal orientation of $\mathbf{B}$. They will be referred to as E_M beams or B_A beams. Similarly, the amplitude function $\mathbf{W}$ [Eq. (2.16b)] results in E_A beams or B_M beams. Since the field vectors of E_M and E_A beams are related by the duality transformation $\mathbf{E}\rightarrow\mathbf{B}$, $\mathbf{B}\rightarrow-\mathbf{E}$, we treat below only the E_M beams.

2. Amplitude functions for weak gravitational fields

The gravitational fields are governed by the nonlinear Einstein equations, which can be linearized in the case of weak fields [18]. Let

$$g_0^{-1}=\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_3-\mathbf{e}_4\otimes\mathbf{e}_4 \tag{2.17}$$

be the twice contravariant metric tensor of free space, where $\otimes$ is the tensor product. A weak gravitational wave can be treated as a small variation h of the metric tensor $g^{-1}=g_0^{-1}+h$. For each plane weak gravitational wave, there exists a reference frame in which this wave is transverse [18]. A transverse wave with the wave vector $\mathbf{k}=k\mathbf{e}_r$ satisfies the conditions $h\cdot\mathbf{e}_r=h\cdot\mathbf{e}_4=0$ and $\mathbf{e}_r\cdot h=\mathbf{e}_4\cdot h=0$. It is described by a symmetric tensor h with zero trace ($h_t=0$) and has two independent polarization states given by

$$h_1(\theta,\varphi)=\tfrac{1}{2}(\mathbf{e}_{\theta'}\otimes\mathbf{e}_{\theta'}-\mathbf{e}_{\varphi}\otimes\mathbf{e}_{\varphi}), \tag{2.18a}$$

$$h_2(\theta,\varphi)=\tfrac{1}{2}(\mathbf{e}_{\theta'}\otimes\mathbf{e}_{\varphi}+\mathbf{e}_{\varphi}\otimes\mathbf{e}_{\theta'}). \tag{2.18b}$$

To obtain orthonormal gravitational beams, we use the amplitude functions

$$h_{\pm}(\theta,\varphi)=h_1\pm ih_2=\tfrac{1}{2}(\mathbf{e}_{\theta'}\pm i\mathbf{e}_{\varphi})\otimes(\mathbf{e}_{\theta'}\pm i\mathbf{e}_{\varphi}). \tag{2.19}$$

These amplitudes satisfy the relations

$$(h_{\pm}^{\dagger}Q_{\pm}h_{\pm})_t=\cos\theta',\quad (h_{\pm}^{\dagger}Q_{\pm}h_{\mp})_t=0 \tag{2.20}$$

with $Q_{\pm}=\pm i\mathbf{e}_3^{\times}$.

C. Quasimonochromatic beams

In this paper, we also treat beams $\breve{\mathbf{W}}_j^s(\mathbf{r},t)$ with three-dimensional beam manifold $\mathcal{B}_3=\mathcal{B}\times[\omega_-,\omega_+]$, related with $\mathbf{W}_j^s(\mathbf{r},t)$ [Eq. (2.7)] as

$$\breve{\mathbf{W}}_j^s(\mathbf{r},t)=\frac{1}{2\Delta\omega}\int_{\omega_-}^{\omega_+}\mathbf{W}_j^s(\mathbf{r},t)d\omega, \tag{2.21}$$

where $\Delta\omega=(\omega_+-\omega_-)/2$. In the case of quasimonochromatic beams, $\Delta\omega\ll\omega$.

The function $\mathbf{W}(\theta,\varphi)$ is frequency-independent. If the beam state function $\nu(\theta,\varphi)$ also is frequency-independent, or its frequency dependence is negligibly small, we have

$$\breve{\mathbf{W}}_j^s(\mathbf{r},t)=e^{-i\omega t}\int_0^{2\pi}d\varphi\int_{\theta_1}^{\theta_2}e^{i\mathbf{r}\cdot\mathbf{k}(\theta,\varphi)}Y_j^s(\theta,\varphi)\times j_0\{p_0[\mathbf{r}\cdot\mathbf{k}(\theta,\varphi)-\omega t]\}\times\nu(\theta,\varphi)\mathbf{W}(\theta,\varphi)\sin\theta\,d\theta, \tag{2.22}$$

where $\omega=(\omega_++\omega_-)/2$ and $p_0=\Delta\omega/\omega$. Hence, the beam is formed from plane-wave packets moving with the light velocity c. Upon integrating $\breve{\mathbf{W}}_j^s(\mathbf{r},t)$ with respect to spatial coordinates, we obtain the norm

$$\|\breve{\mathbf{W}}_j^s\|\equiv\int\breve{\mathbf{W}}_j^{s\dagger}(\mathbf{r},t)\breve{\mathbf{W}}_j^s(\mathbf{r},t)dV=\frac{4\pi^3c^3}{\kappa_0\omega^2\Delta\omega}\int_0^{2\pi}d\varphi\int_{\theta_1}^{\theta_2}|Y_j^s(\theta,\varphi)\nu(\theta,\varphi)|^2\times\frac{\sin^2\theta}{\sin(\kappa_0\theta)}\mathbf{W}^{\dagger}(\theta,\varphi)\mathbf{W}(\theta,\varphi)d\theta. \tag{2.23}$$

For the beams under consideration, this norm is finite. In particular, for electromagnetic fields with the amplitude functions $\mathbf{W}$ [Eqs. (2.16)], we have $\mathbf{W}^{\dagger}\mathbf{W}=|\mathbf{E}|^2+|\mathbf{B}|^2$, i.e., $\|\breve{\mathbf{W}}_j^s\|$ is proportional to the total energy of the field.

III. ORTHONORMAL ELECTROMAGNETIC BEAMS

In this section, we consider the time-harmonic beams $\mathbf{W}_j^s$ [Eq. (2.7)] with $\theta_1=0$ and $\theta_2\leqslant\pi/2$.

A. Beams with $\Omega=2\pi$ and $\kappa_0=1$

To obtain families of orthonormal beams with $\Omega=2\pi$, one can set $\theta_1=0$, $\theta_2=\pi/2$, and $\kappa_0=1$. In this case, the beam manifold $\mathcal{B}$ is the northern hemisphere S_N^2 of S^2, and the function $\nu=\nu(\theta,\varphi)$ reduces to a constant. Electric and magnetic fields as well as energy parameters of such orthonormal beams are found in explicit form in Ref. [7]. In particular, it is shown that time average energy densities w_e and w_m of electric and magnetic fields, and Poynting's vector $\mathbf{S}$, can be written as

$$w_e=w_0w_M,\quad w_m=w_0w_A,\quad w_0=S_0/c, \tag{3.1}$$

$$\mathbf{S}=S_0(S_R'\mathbf{e}_R+S_A'\mathbf{e}_A+S_N'\mathbf{e}_3),\quad S_0=N_Q/\lambda^2, \tag{3.2}$$

where

$$\mathbf{e}_R=\mathbf{e}_1\cos\psi+\mathbf{e}_2\sin\psi, \tag{3.3a}$$

$$\mathbf{e}_A=-\mathbf{e}_1\sin\psi+\mathbf{e}_2\cos\psi, \tag{3.3b}$$

$$\mathbf{r}=R\mathbf{e}_R+z\mathbf{e}_3,\quad R=r\sin\gamma,\quad z=r\cos\gamma, \tag{3.3c}$$

and w_e, w_m, S_R', S_A', and S_N' are independent of the azimuthal angle ψ. Some energy characteristics, such as the

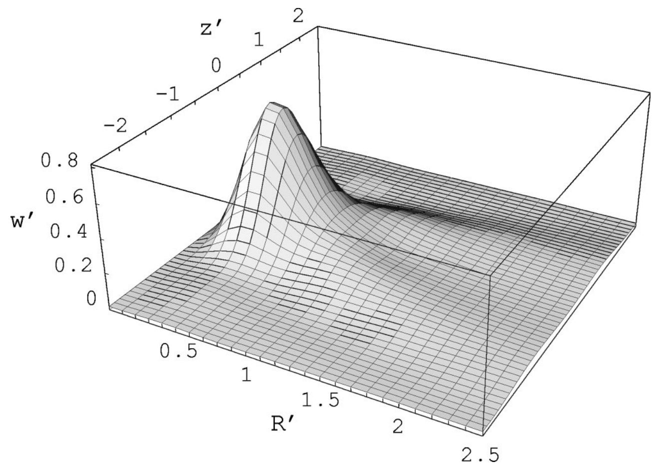

FIG. 1. Normalized energy density $w' = w_M + w_A$; $R' = R/\lambda$; $z' = z/\lambda$; $\Omega = 2\pi$; $j = s = 2$.

dependence of S'_N and S'_A on R at the plane $z = 0$, are presented in Ref. [7]. However, to gain a better insight into the unique properties of these beams, an analysis of the spatial distributions of energy density and energy fluxes is needed.

To illustrate the spatial distribution of the normalized energy density $w' = w_M + w_A$, it is sufficient to calculate values of w' in a meridional plane. High energy density in a very small core region (see Fig. 1) is a distinguishing feature of the fields under consideration.

For the beams defined by the zonal spherical harmonics $(s = 0)$, $S'_A \equiv 0$ [7]. Lines of energy flux for such beams lie in meridional planes (see Figs. 2 and 3). For the beams with $s \neq 0$, lines of energy flux have twisting and spiral forms (see Fig. 4). Such localized fields can be described as a kind of electromagnetic ''tornados.''

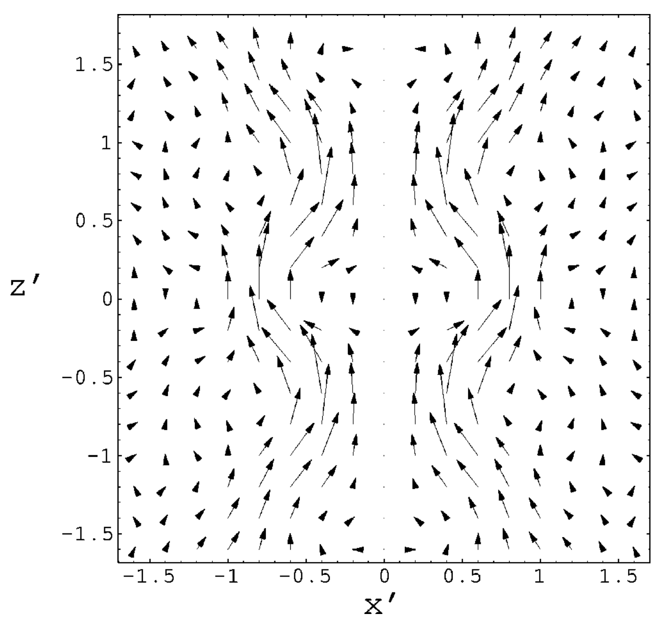

FIG. 2. Poynting's vector field in the plane $x^2 = 0$; $\Omega = 2\pi$; $j = 2$, $s = 0$; $x' = x^1/\lambda$; $z' = x^3/\lambda$.

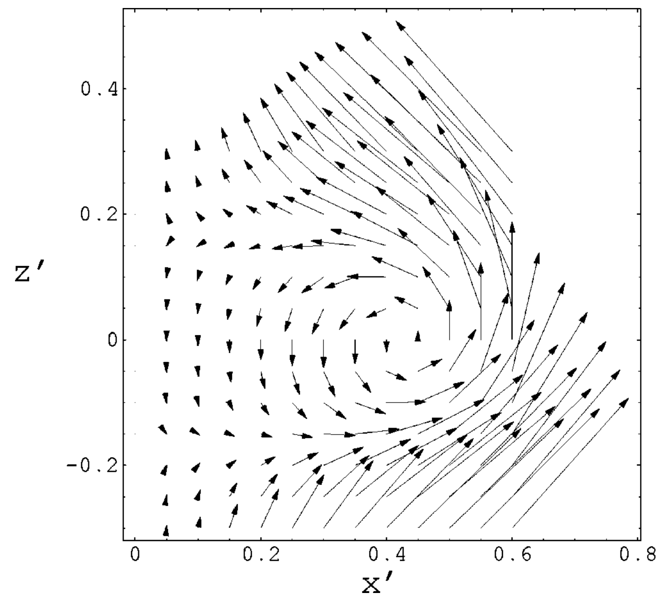

FIG. 3. Poynting's vector field with whirl structure; the parameters of the beam and the notations are the same as for Fig. 2.

B. Beams with $\Omega \leqslant 2\pi$ and $\kappa_0 \leqslant 1/2$

The beam base used above results in two different sets of orthonormal beams defined by the spherical harmonics Y_j^s with even and odd j, respectively. But it may be advantageous to obtain a complete system of orthonormal beams $\mathbf{W}_j^s$, defined by the whole set of spherical harmonics Y_j^s, for which $\langle \mathbf{W}_j^s | Q | \mathbf{W}_{j'}^{s'} \rangle = 0$ if at least one of the three conditions is met: $j' \neq j$, $s' \neq s$, or the beams have the alternative polarization states (E_M and E_A beams).

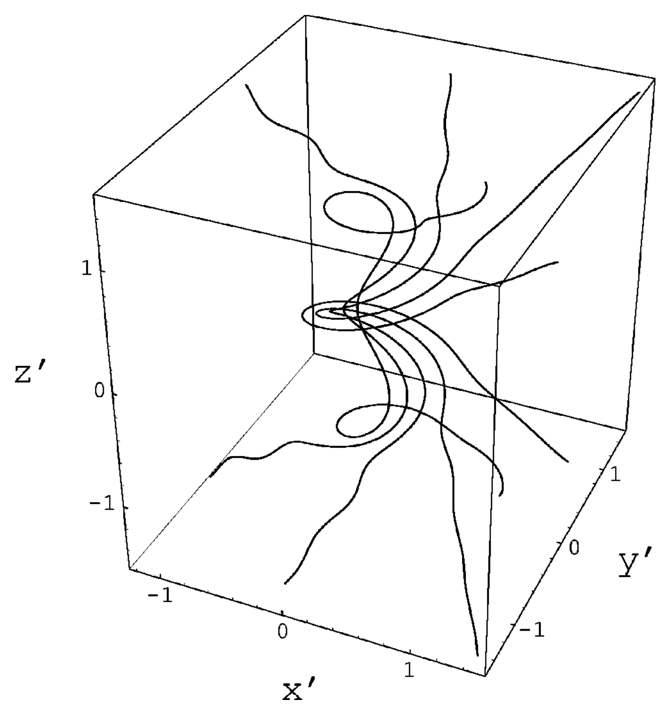

FIG. 4. Lines of energy flux; $\Omega = 2\pi$; $j = s = 2$; $x' = x^1/\lambda$; $y' = x^2/\lambda$; $z' = x^3/\lambda$.

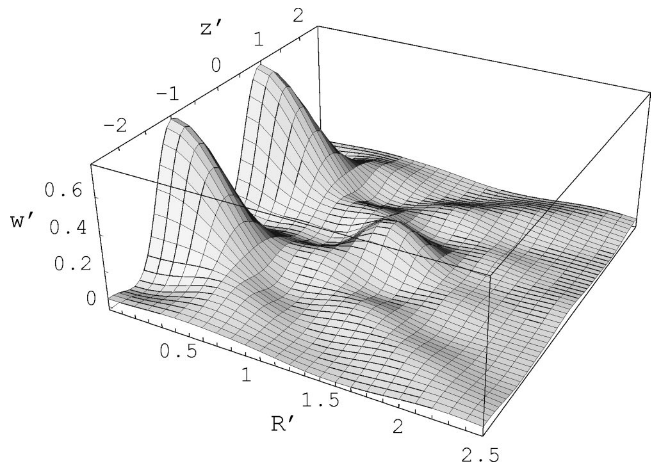

FIG. 5. Normalized energy density $w'=w_M+w_A$; $R'=R/\lambda$; $z'=z/\lambda$; $\Omega=2\pi$; $\kappa_0=1/2$; $j=2$, $s=0$.

To this end, let us set the beam base by Eqs. (2.12)–(2.16) with $\theta_1=0$, $\theta_2=\pi$, and $\kappa_0\leqslant\frac{1}{2}$. In this case, the beam manifold is the unit sphere ($\mathcal{B}=S^2$), $\Omega=2\pi(1-\cos\kappa_0\pi)\leqslant 2\pi$, and

$$\nu(\theta)=\frac{2}{\lambda}\sqrt{\frac{2\pi\kappa_0 N_Q\sin(\kappa_0\theta)}{c\sin\theta}}. \tag{3.4}$$

The smaller κ_0 is, the smaller is Ω, i.e., the beam becomes more collimated. Conversely, if $\kappa_0=1/2$, i.e., $\Omega=2\pi$, the beam has a pronounced core region (see Fig. 5). When $s\neq 0$ and $\kappa_0=\frac{1}{2}$, or $\kappa_0\approx\frac{1}{2}$, such beams also resemble electromagnetic tornados with spiral energy fluxes.

IV. ELECTROMAGNETIC STORMS, WHIRLS, AND TORNADOS

Let us briefly outline unique properties of time-harmonic localized fields $\mathbf{W}_j^s$ [Eq. (2.7)] with $\theta_1=0$, $\pi/2\leqslant\theta_2\leqslant\pi$, and $\kappa_0=1$, i.e., with $\theta'=\theta$ and $2\pi\leqslant\Omega\leqslant 4\pi$. For the sake of simplicity, we assume that the beam state function $\nu=\nu(\theta,\varphi)$ reduces to a constant. A set of these exact time-harmonic solutions of the free-space homogeneous Maxwell equations consists of three subsets—''storms,'' ''whirls,'' and ''tornados''—for which time average energy flux is identically zero at all points, azimuthal and spiral, respectively.

If $\theta_2=\pi$, $\mathcal{B}=S^2$, and $\Omega=4\pi$, the fields under consideration are formed from plane waves of all possible propagation directions. They are in effect three-dimensional standing waves with a rather involved structure of interrelated electric and magnetic fields [6–9].

For E_A and B_A electromagnetic storms, both of which are defined by the zonal spherical harmonics ($s=0$), the time average Poynting vector $\mathbf{S}$ is vanishing at all points [6–9]. The electric field $\mathbf{E}$ of E_A storms has the only nonvanishing component (azimuthal), whereas the azimuthal component of the magnetic field $\mathbf{B}$ is everywhere zero. The opposite situation occurs with B_A storms.

The spherical harmonics with $s\neq 0$ define electromagnetic whirls for which the time average Poynting vector $\mathbf{S}$ has the only nonvanishing component (azimuthal), i.e., $\mathbf{S}=S_0 S_A'\mathbf{e}_A$ [6–9]. This component, as well as the energy densities w_e and w_m of the electric and magnetic fields, is independent of the azimuthal angle ψ. The whirls with $j>s\geqslant 1$ have two major domains (above and below the equatorial plane) with large energy fluxes [8]. The whirls with $j=s\geqslant 1$ have only one such domain, and the energy flux peaks in the equatorial plane.

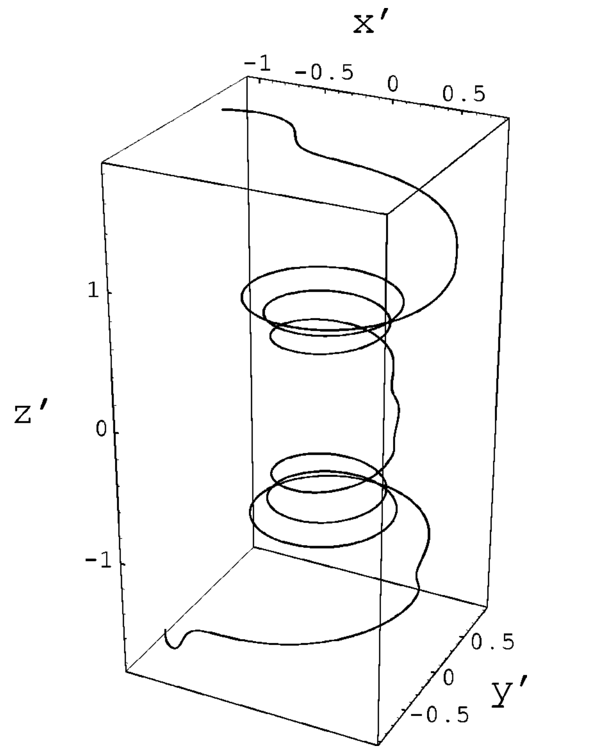

FIG. 6. Line of energy flux; $\theta_2=5\pi/6$; $j=4$, $s=2$; $x'=x^1/\lambda$; $y'=x^2/\lambda$; $z'=x^3/\lambda$.

Let us now consider a family of fields $\mathbf{W}_j^s$ [Eq. (2.7)] with $\theta_1=0$, $\pi/2<\theta_2<\pi$, and $\kappa_0=1$, i.e., with $\theta'=\theta$ and $2\pi<\Omega<4\pi$. Similar to storms and whirls, these fields are highly localized. However, the normal and radial components of time average Poynting's vector $\mathbf{S}$ are not vanishing. As a result, lines of energy flux become spiral (see Fig. 6), provided that $s\neq 0$. Figure 6 shows a typical energy flux line of such a field. We refer to these unique localized fields with spiral energy flux lines as electromagnetic tornados. They bear some similarities to the fields treated in Sec. III A, but their lines of energy flux more closely resemble spirals. As θ_2 tends to π, the step of these spirals decreases.

For the fields with $s=0$, $\theta_1=0$, $\pi/2<\theta_2<\pi$, and $\kappa_0=1$, the lines of energy flux lie in meridional planes. These fields are intermediate in properties between the electromagnetic storms and the beams with $s=0$ and $\Omega=2\pi$ (see Sec. III A).

V. LOCALIZED WEAK GRAVITATIONAL FIELDS

Electromagnetic and weak gravitational plane waves are described by antisymmetric F and symmetric h four-dimensional field tensors, respectively. Both tensors have zero trace. For each family of electromagnetic fields treated in the previous sections, there exists a similar family of weak gravitational fields defined by the same spherical harmonics. Some significant features, such as the localization of field oscillations, are characteristic for both fields. In this section, we present two types of localized weak gravitational fields in the source free space, defined by Eqs. (2.7), (2.18), and (2.19) with $\kappa_0 = 1$ $(\theta' = \theta)$.

A. Orthonormal gravitational beams with $\Omega = 2\pi$

To obtain orthonormal gravitational beams with $\Omega = 2\pi$, we set $\theta_1 = 0$, $\theta_2 = \pi/2$, $\kappa_0 = 1$, and replace $\mathbf{W}(\theta,\varphi)$ [Eq. (2.7)] by $h_\pm(\theta,\varphi)$ [Eq. (2.19)]. In this case, the function ν [Eq. (2.6)] reduces to a constant, and the beams are defined by the real parts of complex tensor functions

$$\begin{aligned} h_\pm(\mathbf{x}) &= h_\pm(\mathbf{r},t) \\ &= \nu_4 e^{-i\omega t}\int_0^{2\pi} d\varphi \int_0^{\pi/2} e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)} Y_j^s(\theta,\varphi) \\ &\quad \times h_\pm(\theta,\varphi)\sin\theta\, d\theta \\ &= \frac{\nu_4}{2} e^{i(s\psi-\omega t)}\{\rho I_j^{ss-2}[1+\cos^2 \pm 2\cos] \\ &\quad + \rho^* I_j^{ss+2}[1+\cos^2 \mp 2\cos] \\ &\quad - \rho_2 I_j^{ss-1}[\sin\circ 2 \pm 2\sin] \\ &\quad - \rho_2^* I_j^{ss+1}[\sin\circ 2 \mp 2\sin] + (\rho_3-\rho_1) I_j^{ss}[\sin^2]\}, \end{aligned} \tag{5.1}$$

where

$$\rho = \mathbf{e}\otimes\mathbf{e}, \quad \rho_1 = \mathbf{e}\otimes\mathbf{e}^* + \mathbf{e}^*\otimes\mathbf{e} = \tfrac{1}{2}(\mathbb{1}-\rho_3), \tag{5.2a}$$

$$\rho_2 = \tfrac{1}{2}(\mathbf{e}\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}), \quad \rho_3 = \mathbf{e}_3\otimes\mathbf{e}_3, \tag{5.2b}$$

$$\mathbf{e} = (\mathbf{e}_R + i\mathbf{e}_A)/2, \quad \nu_4 = \frac{1}{\lambda}\sqrt{2N_Q}. \tag{5.3}$$

Complex scalar functions $I_j^{sm}[f] = I_j^{sm}[f](r,\gamma)$ are defined by the spherical harmonics Y_j^s, an integer m, and a scalar function $f = f(\theta)$. The definitions and the properties of these functions are presented in Ref. [7]. The above notations emphasize the fact that $I_j^{sm}[f](r,\gamma)$ at fixed r and γ are functionals regarding *f*. For any given *f*, $I_j^{sm}[f](r,\gamma)$ is a function of r and γ. When it will not cause a misunderstanding, we omit the arguments (r,γ) and/or $[f]$. The real and imaginary parts of I_j^{sm} can be separated as [7]

$$I_j^{sm} = i^{|m|}(J_{j0}^{sm} + iJ_{j1}^{sm}). \tag{5.4}$$

The intensity of metric oscillations is characterized by the norm $w = (hh^\dagger)_t$. Hence, we obtain

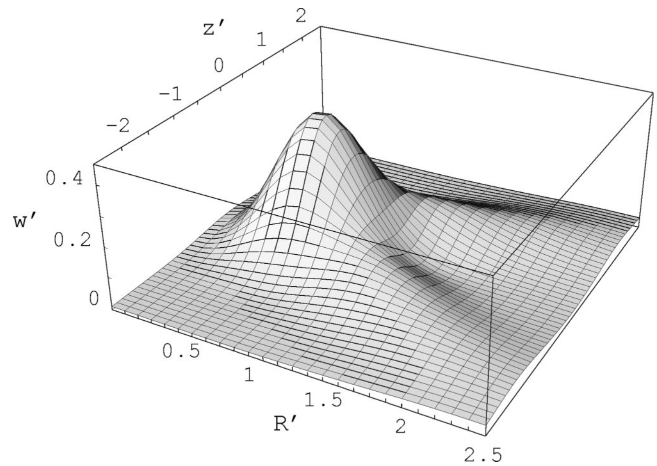

FIG. 7. Normalized intensity w' of metric oscillations $w' = 2w_-/\nu_4^2$; $R' = R/\lambda$; $z' = z/\lambda$; $\Omega = 2\pi$; $j = 3$; $s = 2$.

$$\begin{aligned} w_\pm &= (h_\pm^\dagger h_\pm)_t \\ &= \frac{\nu_4^2}{16}\sum_{p=0}^{1}\{(J_{jp}^{ss-2}[1+\cos^2 \pm 2\cos])^2 \\ &\quad + (J_{jp}^{ss+2}[1+\cos^2 \mp 2\cos])^2 + (J_{jp}^{ss-1}[\sin\circ 2 \pm 2\sin])^2 \\ &\quad + (J_{jp}^{ss+1}[\sin\circ 2 \mp 2\sin])^2 + 6(J_{jp}^{ss}[\sin^2])^2\}. \end{aligned} \tag{5.5}$$

Naturally, the normalization constant N_Q [Eq. (2.3)] must satisfy the condition $w_\pm \ll 1$. Then we obtain

$$\langle h_\pm | Q_\pm | h_\pm\rangle = \int_{\sigma_0} (h_\pm^\dagger Q_\pm h_\pm)_t d\sigma_0 = N_Q, \tag{5.6}$$

where the integrand is given by

$$\begin{aligned} (h_\pm^\dagger Q_\pm h_\pm)_t = \pm\frac{\nu_4^2}{8}\sum_{p=0}^{1}\{&(J_{jp}^{ss-2}[1+\cos^2 \pm 2\cos])^2 \\ &- (J_{jp}^{ss+2}[1+\cos^2 \mp 2\cos])^2 + \tfrac{1}{2}(J_{jp}^{ss-1}[\sin\circ 2 \\ &\pm 2\sin])^2 - \tfrac{1}{2}(J_{jp}^{ss+1}[\sin\circ 2 \mp 2\sin])^2\}. \end{aligned} \tag{5.7}$$

If $s = 0$, the latter reduces to

$$\begin{aligned} (h_\pm^\dagger Q_\pm h_\pm)_t = \nu_4^2 \sum_{p=0}^{1}\{&J_{jp}^{02}[\cos] J_{jp}^{02}[1+\cos^2] \\ &+ \tfrac{1}{2} J_{jp}^{01}[\sin] J_{jp}^{01}[\sin\circ 2]\}, \end{aligned} \tag{5.8}$$

and $w_+ = w_-$. For the beams h_- and h_+, defined by the spherical harmonic Y_3^2, the intensity of metric oscillations is illustrated in Figs. 7 and 8.

Orthonormal gravitational beams with $\Omega \leq 2\pi$ and $\kappa_0 \leq \frac{1}{2}$ can be obtained by using a beam state function that differs from $\nu(\theta)$ [Eq. (3.4)] only by a constant factor.

B. Gravitational whirls and storms

Let us now set $\theta_1 = 0$, $\theta_2 = \pi$, $\kappa_0 = 1$, and define the amplitude function by Eqs. (2.18), assuming that the beam state

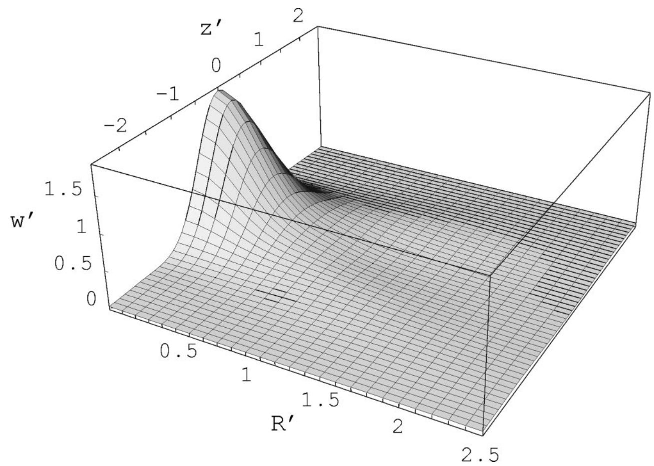

FIG. 8. Normalized intensity w' of metric oscillations $w' = 2w_+/\nu_4^2$; $R' = R/\lambda$; $z' = z/\lambda$; $\Omega = 2\pi$; $j=3$; $s=2$.

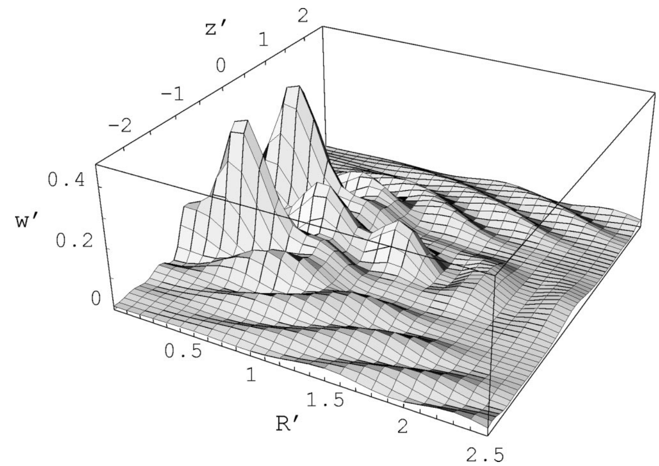

FIG. 9. Normalized intensity w' of metric oscillations $w' = 2w_1/\nu_4^2$; $R' = R/\lambda$; $z' = z/\lambda$; $\Omega = 4\pi$; $j=3$; $s=1$.

function ν reduces to a constant. This results in two sets of weak gravitational fields that are defined by the real parts of the complex tensor functions

$$h_n(\mathbf{x}) \equiv h_n(\mathbf{r},t) = \nu_4 e^{-i\omega t}\int_0^{2\pi} d\varphi \int_0^{\pi} e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)} \times Y_j^s(\theta,\varphi) h_n(\theta,\varphi)\sin\theta\, d\theta, \tag{5.9}$$

where $n = 1,2$, and the scalar coefficient ν_4 specifies the amplitudes of partial plane waves ($\nu_4 \ll 1$). Substitution of h_1 and h_2 [Eq. (2.18)] in Eq. (5.9) yields

$$\begin{aligned} h_1 = \nu_4 i^{|s|+p} e^{i(s\psi - \omega t)} \{ & \rho\alpha(s) J_{jp}^{ss-2}[1+\cos^2] \\ & + \rho^* \alpha(-s) J_{jp}^{ss+2}[1+\cos^2] - \rho_2(-1)^q \beta(-s) \\ & \times J_{jq}^{ss-1}[\sin\circ 2] - \rho_2^*(-1)^q \beta(s) J_{jq}^{ss+1}[\sin\circ 2] \\ & + (\rho_3 - \rho_1) J_{jp}^{ss}[\sin^2]\}, \end{aligned} \tag{5.10a}$$

$$\begin{aligned} h_2 = 2\nu_4 i^{|s|+p} e^{i(s\psi - \omega t)} \{ & \rho\alpha(s)(-1)^p J_{jq}^{ss-2}[\cos] \\ & - \rho^* \alpha(-s)(-1)^p J_{jq}^{ss+2}[\cos] \\ & - \rho_2 \beta(-s) J_{jp}^{ss-1}[\sin] + \rho_2^* \beta(s) J_{jp}^{ss+1}[\sin]\}, \end{aligned} \tag{5.10b}$$

where

$$\beta(s) = \begin{cases} -1 & (s = -1,-2,\dots) \\ 1 & (s = 0,1,2,\dots), \end{cases} \tag{5.11}$$

$\alpha(s) = 2\delta_{1s} - 1$, $p = 1 - q = 0$ if $j + |s|$ is even, and $p = 1 - q = 1$ if $j + |s|$ is odd. The traces and the four-dimensional determinants of these tensors are vanishing at all points, and $\mathbf{e}_4 \cdot h_n = h_n \cdot \mathbf{e}_4 \equiv 0$, $n = 1,2$.

The intensities of metric oscillations for these fields are given by

$$\begin{aligned} w_1 = \frac{\nu_4^2}{4} \{ & (J_{jp}^{ss-2}[1+\cos^2])^2 + (J_{jp}^{ss+2}[1+\cos^2])^2 \\ & + (J_{jq}^{ss-1}[\sin\circ 2])^2 + (J_{jq}^{ss+1}[\sin\circ 2])^2 + 6(J_{jp}^{ss}[\sin^2])^2\}, \end{aligned} \tag{5.12a}$$

$$\begin{aligned} w_2 = \nu_4^2 \{ & (J_{jq}^{ss-2}[\cos])^2 + (J_{jq}^{ss+2}[\cos])^2 \\ & + (J_{jp}^{ss-1}[\sin])^2 + (J_{jp}^{ss+1}[\sin])^2\}. \end{aligned} \tag{5.12b}$$

Figures 9 and 10 illustrate the intensity of the metric oscillations in the core regions of the whirls h_1 and h_2, defined by the spherical harmonic Y_3^1.

Gravitational storms are defined by the zonal spherical harmonics as

$$\begin{aligned} h_1 = \frac{\nu_4}{2} i^p e^{-i\omega t} \{ & (\mathbf{e}_A \otimes \mathbf{e}_A - \mathbf{e}_R \otimes \mathbf{e}_R) J_{jp}^{02}[1+\cos^2] \\ & - (\mathbf{e}_R \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_R)(-1)^q J_{jq}^{01}[\sin\circ 2] \\ & + 2(\rho_3 - \rho_1) J_{jp}^{00}[\sin^2]\}, \end{aligned} \tag{5.13a}$$

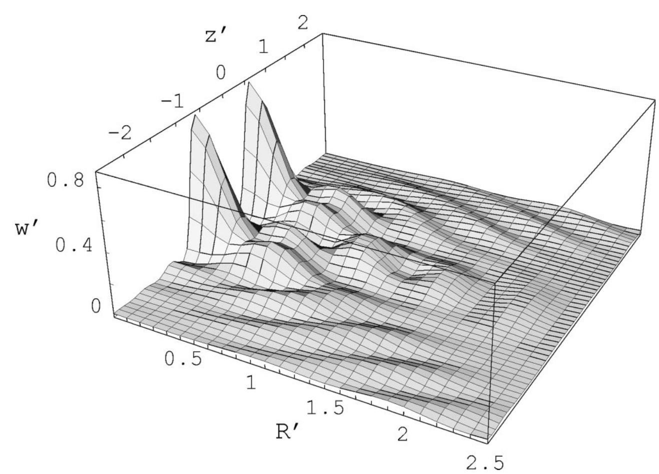

FIG. 10. Normalized intensity w' of metric oscillations $w' = 2w_2/\nu_4^2$; $R' = R/\lambda$; $z' = z/\lambda$; $\Omega = 4\pi$; $j=3$; $s=1$.

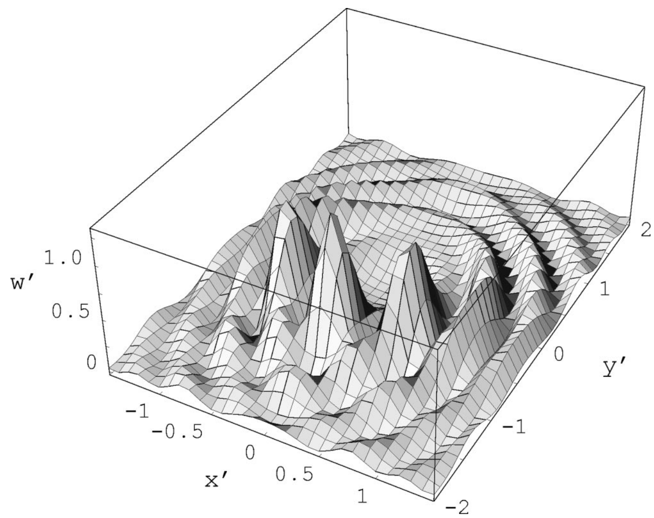

FIG. 11. Normalized instantaneous energy density w' of an electromagnetic whirl moving with the velocity $\mathbf{V}=0.4\,c\,\mathbf{e}_{1'}$ with respect to the frame L'; $j=4$; $s=2$; $x'=x^{1'}/\lambda$; $y'=x^{2'}/\lambda$; $x^{3'}=0$.

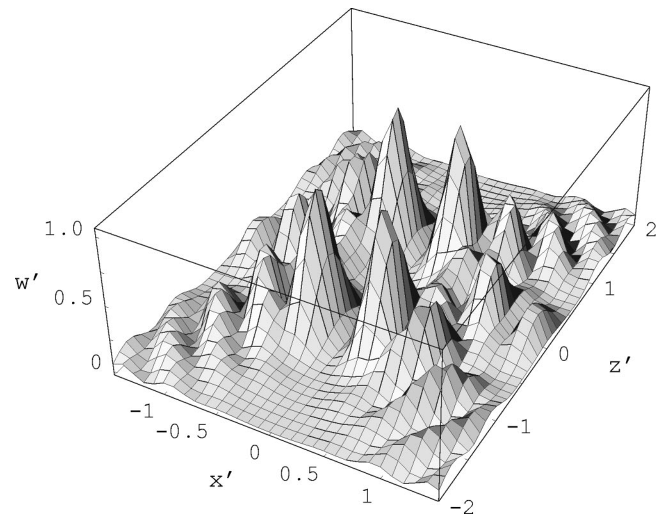

FIG. 12. Normalized instantaneous energy density w' of an electromagnetic whirl moving with the velocity $\mathbf{V}=0.4\,c\,\mathbf{e}_{1'}$ with respect to the frame L'; $j=4$; $s=2$; $x'=x^{1'}/\lambda$; $z'=x^{3'}/\lambda$; $x^{2'}=0$.

$$h_2=\nu_4 i^{-q}e^{-i\omega t}\{(\mathbf{e}_R\otimes\mathbf{e}_A+\mathbf{e}_A\otimes\mathbf{e}_R)(-1)^p J^{02}_{jq}[\cos] + (\mathbf{e}_A\otimes\mathbf{e}_3+\mathbf{e}_3\otimes\mathbf{e}_A)J^{01}_{jp}[\sin]\}. \tag{5.13b}$$

VI. MOVING AND EVOLVING STORMS, WHIRLS, AND TORNADOS

The localized electromagnetic and gravitational fields discussed above are time-harmonic in the Lorentz reference frame L with basis $(\mathbf{e}_i)$. In any other Lorentz frame L' with basis $(\mathbf{e}'_i)$, they will be observed as a kind of electromagnetic [7] or gravitational missile moving without dispersing at speed $V<c$. Because of the very involved dependence of complex fields $\mathbf{E}$, $\mathbf{B}$, or h on time and spatial coordinates, parameters $|\mathbf{E}|^2$, $|\mathbf{B}|^2$, or $w=(hh^\dagger)_t$ provide only a rough idea of the missile field structure [7], whereas instantaneous values of real fields give an accurate picture. Figures 11 and 12 illustrate both the localization of field oscillations and the asymmetry (Fig. 11) caused by the movement of whirls.

The time-harmonic solutions presented in the previous sections can be applied as reasonably accurate models of real physical fields. A more realistic description can be achieved by integrating these solutions with respect to frequency [7]. As illustration, let us consider basic properties of quasimonochromatic fields defined by Eqs. (2.21) and (2.22). To be specific, we discuss below evolving electromagnetic whirls, but other evolving electromagnetic and weak gravitational fields (storms and tornados) can be treated similarly.

The field $\breve{\mathbf{W}}^s_j(\mathbf{r},t)$ [Eq. (2.22)] is formed from the infinitesimal plane-wave packets with the envelope function

$$j_0(p_0\varphi_0)=\frac{\sin(p_0\varphi_0)}{p_0\varphi_0}, \tag{6.1}$$

where $\varphi_0=\mathbf{r}\cdot\mathbf{k}(\theta,\varphi)-\omega t$. Since $p_0\ll 1$, the field can be described as an evolving whirl in the neighborhood of the point $\mathbf{r}=0$. At $t=0$, the whirl reaches its maximum intensity and closely resembles a time-harmonic whirl. In particular, lines of energy flux are circular for both whirls. At $-\pi/\Delta\omega<t<0$ and $0<t<\pi/\Delta\omega$, the energy flux lines of the evolving whirl are convergent (Fig. 13) and divergent (Fig. 14), respectively. When $t\to\pm\infty$, the field tends to zero at all points $\mathbf{r}$.

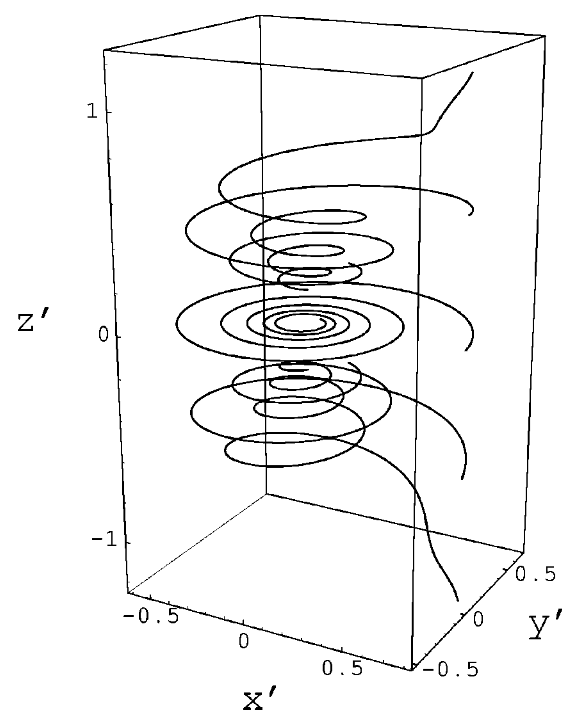

FIG. 13. Convergent lines of energy flux of an evolving electromagnetic whirl; $\Omega=4\pi$; $j=s=4$; $p=0.05$; $ct=-7\lambda$; $x'=x^1/\lambda$; $y'=x^2/\lambda$; $z'=x^3/\lambda$.

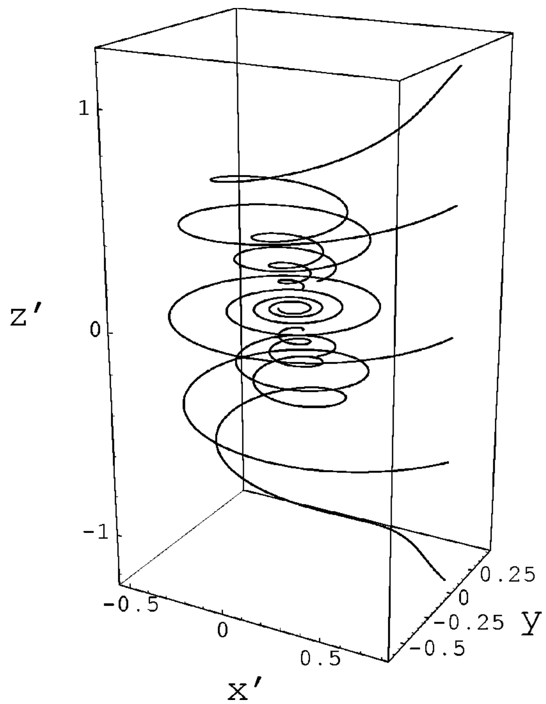

FIG. 14. Divergent lines of energy flux of an evolving electromagnetic whirl; $\Omega=4\pi$; $j=s=4$; $p=0.05$; $ct=8\lambda$; $x'=x^{1'}/\lambda$; $y'=x^{1'}/\lambda$; $z'=x^{3'}/\lambda$.

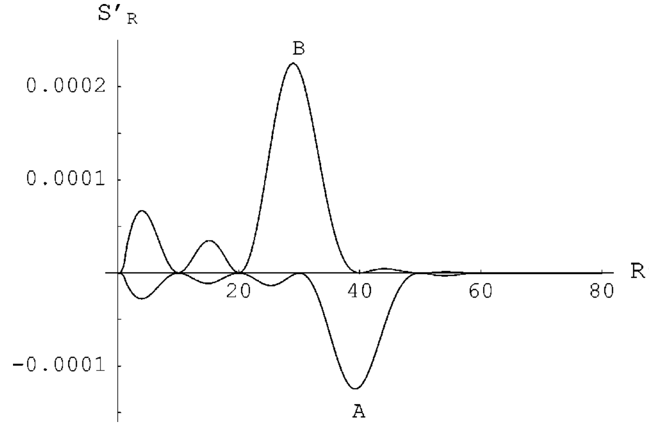

FIG. 15. Radial component S_R' of the normalized energy flux vector as a function of $R'=R/\lambda$; $z=0;j=s=4$; $\Delta\omega/\omega=0.05$; (A) $ct=-40\lambda$; (B) $ct=30\lambda$.

The evolution of the field can be described as follows. A whirl originates at infinity at $t=-\infty$ as an infinitely small converging wave. At $t\leqslant-\pi/\Delta\omega$, there is a very small converging wave with a maximum peak at the distance $r=-ct$ (see curve A in Fig. 15). During all this time, there is also a weak whirl in the neighborhood of the point $\mathbf{r}=0$. It passes through maxima and minima of activity, gradually gaining in intensity as $t\rightarrow0$. Figure 15 illustrates the radial energy fluxes at two different instants of minimum whirl activity. The total field can be described as the superposition of converging and expanding waves with ever-changing proportion. At $t>0$, the whirl, still passing through maxima and minima of activity, gradually transforms into an expanding wave (see curve B in Fig. 15), which vanishes in infinity as $t\rightarrow+\infty$. It follows from Eq. (2.23) that the evolving storms, whirls, and tornados have finite total energy.

VII. CONCLUSION

Unique solutions of wave equations, which describe localized electromagnetic and weak gravitational time-harmonic fields in the source-free space, are obtained using expansions in plane waves. These fields have a very small and clearly defined core region with maximum intensity of field oscillations. Outside the core, the intensity of oscillations rapidly decreases in all directions. Each family of solutions consists of vector or tensor functions that have integral expansions in plane waves propagating in the same given solid angle Ω.

Our main concern in this paper is with the families of orthonormal beams with $\Omega=2\pi$ and the families of three-dimensional standing waves with $\Omega=4\pi$. In addition, some specific electromagnetic fields with $\Omega\leqslant2\pi$ and $2\pi\leqslant\Omega\leqslant4\pi$ are presented.

The peculiarities of energy transport in such fields are illustrated by the example of localized electromagnetic fields. In a given Lorentz frame L, a set of the obtained exact time-harmonic solutions of the free-space homogeneous Maxwell equations consists of three subsets (storms, whirls, and tornados) for which time average energy flux is identically zero at all points, azimuthal and spiral, respectively. In any other Lorentz frame L', they will be observed as a kind of electromagnetic missile moving without dispersing at speed $V<c$.

The properties of evolving fields, obtained by integrating the time-harmonic solutions with respect to frequency, are briefly outlined. Since evolving electromagnetic storms, whirls, tornados, and various types of moving and evolving missiles are described by the exact solutions of Maxwell's equations and have finite total energy, they may exist in nature or can be excited. To this end, the modern antenna technology [19] provides promising tools, such as large array antennas with tens of thousands and even well over 100 000 elements, active integrated antennas, and beam-forming techniques.

In this paper, the localized gravitational fields are treated in the linear approximation. This gives grounds to propose the problem of searching for exact solutions of the Einstein empty space field equations, which reduce, in the case of weak fields, to the solutions presented above. In solving this problem, the evolving weak fields can be used as the initial conditions. To this end, it is necessary to set the initial moment $t_0\leqslant-\pi/\Delta\omega$ and the parameter N_Q in such a way as to obtain a weak converging wave at $t\leqslant t_0$. If N_Q is sufficiently small, the evolving field will remain everywhere weak at any $t>t_0$. However, by decreasing t_0 and increasing N_Q, one can set the initial conditions to search for a nonlinear evolving field that is everywhere weak and can be described by the presented solutions only at $t\leqslant t_0$. The further evolution of this converging wave should be investigated by solving the Einstein equations.

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